radiation sensitivity. The aim of further work along these lines is to find ways of picturing biological effects of radiation and premises to be used in their interpretation.

Summary.—(1) Pepsin-albumin films may be inactivated by small doses of x-rays. Doses of about 100 r produce about 50 per cent inactivation.

(2) The sensitivity to radiation depends on the physical configuration of the molecules. It may be varied by surface compression.

(3) Calculations suggest that the effects of a single radiation event (ionization or radical production) may be spread to include a large number of enzyme molecules.

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NOTES ON INTEGRATION, I

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The theory of integration, because of its central rôle in mathematical analysis and geometry, continues to afford opportunities for serious investigation. The need for extending and rounding out the classical studies of Riemann, Stieltjes and Lebesgue has stimulated considerable interest not only in new aspects of the theory but also in the simplification and perfection of the old. The present series of communications is intended to outline a treatment which, while exploiting fully the possibilities for simplification, will attain a high degree of generality. Among the many contributions to the mathematical literature which have provided material for our handling of the subject we wish to cite above all an important paper of Daniell.[1] In spite of the fact that the basic ideas in the present discussion are the common property of mathematicians, some of our results appear to be novel. If they are, it is because we have chosen to introduce and exploit an adaptation of the concept of an "upper integral," making this the technical foundation of the whole theory.[2]

With Daniell, we assume an initially given *elementary integral* $E(f)$ which is defined for a non-void class $\mathfrak{E}$ of real functions f, called the *elementary functions*, with a fixed abstract set X as domain. Specifically, our basic postulates are the following slight modifications of those given by Daniell:

(1) $\mathfrak{E}$ is a non-void class of real functions f on an arbitrary non-void domain X, such that $\alpha f, f + g$, and $|f|$ are in $\mathfrak{E}$ whenever α is a real number and f and g are both in $\mathfrak{E}$; and E is a real-valued function (or operation) defined over $\mathfrak{E}$, such that

$$E(\alpha f) = \alpha E(f),\ E(f + g) = E(f) + E(g),\ E(|f|) \geqq 0;$$

(2) if f and f_n are in $\mathfrak{E}$ and $|f| \leq \sum_{n=1}^{\infty} |f_n|$, then

$$E(|f|) \leq \sum_{n=1}^{\infty} E(|f_n|).$$

At a certain stage in the development, we shall introduce the further requirement that

(3) if f is in $\mathfrak{E}$, then $\min(1, f) = \frac{1}{2}(1 + f - |1 - f|)$ is in $\mathfrak{E}$.

Technically (1) means that $\mathfrak{E}$ is a vector lattice under its natural ordering and E is a positive linear functional on $\mathfrak{E}$. The rôles played by (2) and (3) will be brought out below.

In many important instances of the general theory, $\mathfrak{E}$ consists of the continuous real functions with compact nucleus[3] on a locally compact space X. Under these conditions (2) and (3) both follow from (1): (3) is evident, while (2) is established by constructing a non-negative elementary function g strictly positive on the nucleus of f and demonstrating the existence of an integer $m = m(\epsilon)$, $\epsilon > 0$, such that $|f| - \epsilon g \leqq \sum_{n=1}^{m} |f_n|$. By way of illustration we cite the Riemann integral on a bounded closed domain in n-space or on an n-dimensional manifold, and the Haar integral on a locally compact topological group.[4] Other important instances of the general theory are constructed from distributions of simple type given in advance on X: in X a class of subsets is distinguished and each distinguished set Y is assumed to bear a non-negative weight or measure, $\mu(Y)$; the real linear lattice combinations of the characteristic functions of distinguished sets are taken as the elementary functions; and the elementary integral is then determined in harmony with the requirement that $E(f) = \mu(Y)$ for the characteristic function f of any distinguished set Y. The detailed construction is not difficult once the appropriate techniques have been provided for relating the data to the requirements of (1) and (2). We mention in particular that the given measure must be additive: if Y, Z, and $Y \cup Z$ are distinguished sets, and Y and Z are disjoint, then $\mu(Y \cup Z) = \mu(Y) + \mu(Z)$. The theory of the Lebesgue-Stieltjes integral illustrates the general concepts which have just been described.

We shall consider the class $\mathfrak{G}$ of all extended-real functions defined on the domain X. The admission of $+\infty$ and $-\infty$ as functional values renders a little awkward the definition of such algebraic combinations of functions as $0 \cdot f, f + g$ and $f - g$. In the present context it is appropriate to let $0 \cdot f, f + g, f - g$ designate (ambiguously!) any function in $\mathfrak{G}$ which assumes the respective values $0 \cdot f(x), f(x) + g(x), f(x) - g(x)$ at every x where the latter quantities are defined. As we shall see, this ambiguity raises no serious obstacles in the sequel.

Over $\mathfrak{G}$ we define an extended-real function N by putting

$$(4) \quad N(f) = \inf \{\lambda;\ \lambda = \sum_{n=1}^{\infty} E(|f_n|),\ |f| \leqq \sum_{n=1}^{\infty} |f_n|, f_n \in \mathfrak{E}\}.$$

For non-negative functions, $N(f)$ has the properties of an upper integral; and for certain non-negative functions it coincides with the integral which is presently to be defined. Thus we already have in our possession the mathematical object upon which our interest is concentrated. The formal definition given in (4) is capable of an informal presentation which brings out clearly and rather simply its intuitive justification. Confining our attention to non-negative functions, we may regard (4) as the condensed description of a measuring process. We may think of $\mathfrak{E}$ as providing a stock of measuring rods, the non-negative elementary functions, by means of which the non-negative functions in $\mathfrak{G}$ are to be gauged. Each measuring rod has a magnitude given by its elementary integral. The basic measuring process consists in choosing from stock such an infinite sequence of measuring rods $f_n = |f_n| \geqq 0$ that by addition they combine to surpass $f = |f| \geqq 0$ in the sense indicated by the inequality $f \leqq \sum_{n=1}^{\infty} f_n$. The real number $\lambda = \sum_{n=1}^{\infty} E(f_n)$ obtained by adding together the magnitudes of the particular measuring rods thus employed is then accepted as an estimate, generally in excess, of the magnitude of f. Repetitions of this basic process furnish successively better estimates, convergent to the quantity $N(f)$, when suitable precautions are taken. It is, of course, conceivable that the basic process will fail to produce any real numbers as estimates of the magnitude of a particular function f—either because the inequality $f \leqq \sum_{n=1}^{\infty} f_n$ cannot be realized, or because it implies the divergence of the series $\sum_{n=1} E(f_n)$. Under these circumstances the formal definition requires that we take $N(f) = +\infty$ in accordance with a well-known convention. The principal properties of the function N, all easily deduced from (4), can be listed as follows:

(5) $0 \leqq N(f) \leqq +\infty$;

(6) $N(\alpha f) = |\alpha| N(f)$ unless $\alpha = 0$ and $N(f) = +\infty$;

(7) $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ implies $N(f) \leqq \sum_{n=1}^{\infty} N(f_n)$;

(8) $N(|f|) = N(f)$.

By specializing (7) we see that $N(f + g) \leqq N(f) + N(g)$, and that $N(f) \leqq N(g)$ whenever $|f| \leqq |g|$. The ambiguity of the expressions αf, $f + g$ does not affect the truth of these relations; nor does (2) enter into their proofs. It is only in deriving the property

(9) when f is elementary, $N(f) = E(|f|)$

that (2) finds any direct application in our theory. Clearly (2) and (9) are equivalent assertions, so that whenever we use (9) in the sequel we also use (2) in an essential though implicit manner.

Henceforth we shall be concerned primarily with that part, $\mathfrak{F}$, of $\mathfrak{G}$ which is characterized by the inequality $N(f) < +\infty$. The expression $N(f - g)$ has in $\mathfrak{F}$ the properties of a pseudo-metric. Hence, if we identify functions f and g for which $N(f - g) = 0$, we can treat $\mathfrak{F}$ as a real normed vector-lattice with N as its norm-function. The detailed discussion involves attention to those functions f, called *null functions*, for which $N(f) = 0$. In this connection it is convenient to introduce also the following definitions: a subset of X is called a *null set* if its characteristic function is a null function; the phrase "*almost everywhere*" signifies "with the exception of the points of a certain null set." By the use of (7) it is easy to show that a function is a null function if and only if it vanishes almost everywhere; that any set covered by a countable family of null sets is a null set; and that every function in $\mathfrak{F}$ is finite almost everywhere. These and similar properties show that the null sets play here a rôle analogous to that played by the sets of measure zero in the Lebesgue theory. The identification of functions in $\mathfrak{F}$ is now seen to remove all ambiguity in the meaning of the expressions αf, $f + g$, $f - g$ since every function in $\mathfrak{F}$ is finite almost everywhere. Once these more or less routine matters are disposed of, we can establish the most important single result[5] concerning $\mathfrak{F}$, namely:

(10) the normed vector space $\mathfrak{F}$ is complete (and hence a Banach space).

The proof will be sketched. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{F}$. Without loss of generality we may suppose that f_n is everywhere finite (otherwise we could modify f_n on a null set so that the resulting function would be everywhere finite) and that $N(f_{n+1} - f_n) \leqq 2^{-n}$ (otherwise we could choose a subsequence which converges with the desired rapidity). The series $|f_1| + \sum_{n=1}^{\infty}|f_{n+1} - f_n|$ has the sum g in $\mathfrak{G}$. Since $N(g) \leqq N(f_1) + \sum_{n=1}^{\infty}N(f_{n+1} - f_n) = N(f_1) + \sum_{n=1}^{\infty}2^{-n} \leqq N(f_1) + 1 < +\infty$ by (7), g is actually in $\mathfrak{F}$. Hence the series above converges almost everywhere, and so also do the series $f_1 + \sum_{n=1}^{\infty}(f_{n+1} - f_n)$ and the sequence $\{f_n\}$. Let f be any function in $\mathfrak{G}$ which is equal to the sum of the latter series or, equivalently, to $\lim_{n\to\infty} f_n$ wherever these quantities are defined: we may, for example, take $f = \limsup_{n\to\infty} f_n$ or $f = \liminf_{n\to\infty} f_n$. We then have $|f| \leqq g$, $N(f) \leqq N(g) < +\infty$, $f \in \mathfrak{F}$. Moreover $|f - f_k| \leqq \sum_{n=k}^{\infty}|f_{n+1} - f_n|$ and $N(f - f_k) \leqq \sum_{n=k}^{\infty}N(f_{n+1} - f_n) \leqq 2^{-k+1}$, so that the Cauchy sequence $\{f_n\}$ has f as its limit in $\mathfrak{F}$. This completes the proof.

On $\mathfrak{F}$ we now define the function F by putting $F(f) = N(f^+) - N(f^-)$ where $f^+ = \frac{1}{2}(|f|) + f) \geqq 0, f^- = \frac{1}{2}(|f| - f) \geqq 0$. Since $|f^+ - g^+| \leqq |f - g|$ and $|f^- - g^-| \leqq |f - g|$, this function is continuous in accordance with the inequalities $|F(f) - F(g)| \leqq |N(f^+) - N(g^+)| + |N(f^-) - N(g^-)| \leqq N(f^+ - g^+) + N(f^- - g^-) \leqq 2N(f - g)$. On $\mathfrak{E} \subset \mathfrak{F}$ we have $F(f) = E(f^+) - E(f^-) = E(f)$ by (9). Let $\mathfrak{L}$ be the closure of $\mathfrak{E}$ in $\mathfrak{F}$, and let L be the contraction of F to $\mathfrak{L}$. A function in $\mathfrak{L}$ is said to be *integrable*, and its *general integral* is taken to be $L(f) = N(f^+) - N(f^-)$. The integrable functions are the functions which can be approximated by elementary functions in the sense of the norm for $\mathfrak{F}$. On the other hand, the results of the preceding paragraph show that every integrable function f is equal almost everywhere to $\limsup_{n\to\infty} f_n$, where $\{f_n\}$ is a suitably chosen sequence of elementary functions. The following assertions concerning the general integral can now be justified:

(11) $\mathfrak{L}$ and L enjoy the properties assumed for $\mathfrak{E}$ and E, respectively, in (1), $\mathfrak{L}$ being a complete vector subspace of $\mathfrak{F}$ with norm $N(f) = L(|f|)$;

(12) the sum of a positive-term series of integrable functions is integrable if and only if the corresponding series of integrals converges (necessarily to the integral of the sum-function).

It is clear that (11) follows from (1) by simple continuity arguments. We observe that (12) may be regarded as a sharpened version of (2) formulated in terms of $\mathfrak{L}$ and L; and also that (12) is a generalized form of the theorem of B. Levi in the Lebesgue theory. A proof of (12) will now be sketched. Let $f = \sum_{n=1}^{\infty} f_n$ where $f_n \geqq 0$ and $f_n \in \mathfrak{L}$. If $f \in \mathfrak{F}$, then $\sum_{n=1}^{m} L(f_n) = L(\sum_{n=1}^{m} f_n) = N(\sum_{n=1}^{m} f_n) \leqq N(f) < +\infty$ so that $\sum_{n=1}^{\infty} L(f_n)$ is a convergent positive-term series. On the other hand, if $\sigma = \sum_{n=1}^{\infty} L(f_n) < +\infty$, we have $N(f) \leqq \sum_{n=1}^{\infty} N(f_n) = \sum_{n=1}^{\infty} L(f_n) < +\infty$ so that $f \in \mathfrak{F}$. Moreover $N(f - \sum_{n=1}^{m} f_n) \leqq \sum_{n=m+1}^{\infty} N(f_n) = \sum_{n=m+1}^{\infty} L(f_n)$ so that $\sum_{n=1}^{m} f_n$ converges in $\mathfrak{L}$ to f and $L(\sum_{n=1}^{m} f_n)$ converges to $L(f)$, with the result that $L(f) = \sigma$. From (12) it is now possible to deduce other convergence theorems which correspond in our general theory to such standard results as the Lebesgue dominated-convergence theorem and the Fatou theorem in the Lebesgue theory. As the classical arguments apply without change to the present situation we do not need to go into detail.

Let Φ_m be the class of all positively homogeneous continuous real functions of m real variables $\lambda_1, \ldots, \lambda_m$. To Φ_m belong the functions $\lambda_1, \ldots, \lambda_m$ and all the linear lattice combinations which can be formed from them. On the other hand, it is known that on the compact set where $|\lambda_1| + \ldots + |\lambda_m| = 1$ any continuous function can be uniformly approximated by such combinations.[6] Hence if $\varphi \in \Phi_m$ we can find such a combination ψ_k that $|\varphi - \psi_k| \leqq 1/k$ on this set. It follows that $|\varphi - \psi_k| \leqq 1/k (|\lambda_1| + \ldots + |\lambda_m|)$. Since $\psi_k(f_1, \ldots, f_m)$ is integrable when $f_1, \ldots, f_m$ are, as we noted in (11), and since

$$N(\varphi(f_1, \ldots, f_m) - \psi_k(f_1, \ldots, f_m)) \leqq (N(f_1) + \ldots + N(f_m))/k \rightarrow 0$$

when $k \rightarrow \infty$, we conclude that $\varphi(f_1, \ldots, f_m)$ is integrable. The dominated-convergence theorem enables us to extend this result to the positively homogeneous Baire functions:

(13) if φ is a positively homogeneous real function of m real variables $\lambda_1, \ldots, \lambda_m$ whose contraction to the set $|\lambda_1| + \ldots + |\lambda_m| = 1$ is a bounded Baire function, and if $f_1, \ldots, f_m$ are integrable, then $\varphi(f_1, \ldots, f_m)$ is integrable.

In order to rid ourselves of the restriction to homogeneous functions, we now assume that (3) holds.[7] We find that (13) can be replaced by:

(14) if φ is a finite, not necessarily bounded Baire function of m real variables $\lambda_1, \ldots, \lambda_m$ such that $\varphi(0, \ldots, 0) = 0$, and if $f_1, \ldots, f_m, g$ are integrable functions such that $|\varphi(f_1, \ldots, f_m)| \leqq |g|$, then $\varphi(f_1, \ldots, f_m)$ is integrable.

Indeed, if the constant function everywhere equal to 1 is in $\mathfrak{C}$ or in $\mathfrak{L}$, we can also eliminate the condition $\varphi(0, \ldots, 0) = 0$. By further application of the dominated-convergence theorem we can extend (13) and (14) to functions φ of infinitely many variables.

The principal processes for manipulating the general integral have now been justified. In our second note we shall review some of their applications and consequences.

[1] Daniell, P. J., "A General Form of Integral," *Ann. Math.*, **19**, 279–294 (1917–1918).

[2] Some remarks of H. Blumberg, *Am. Math. Monthly*, **53**, 189 (1946), on Lebesgue measure shed much light on the subject for me and started a train of thought culminating in the introduction of an "upper integral" here.

[3] The nucleus of a function is the closure of the set of points where it assumes non-zero values.

[4] Weil, A., *L'Intégration dans les Groupes Topologiques et ses Applications*, Paris, 1938, 34–38.

[5] This result is believed to be new.

[6] Stone, M. H., "The Generalized Weierstrass Approximation Theorem," *Math. Mag.* **21**, 167–183 (1948); see particularly §2, where Corollary 2 to Theorem 3 gives the relevant information.

[7] Other conditions leading to (14) have been investigated by Mr. H. Rubin, who, as a member of one of my classes, made many useful comments on the subject-matter of this whole paragraph.

ON THE DIFFERENTIAL EQUATIONS OF SLIP FLOW

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In a compressible medium, let the stress tensor be T^i_j, the specific internal energy ϵ, the specific entropy s, the specific volume σ. Then the *mean pressure* p_m and the *pressure* p are given by the definitions

$$p_m \equiv -\frac{1}{3}T^i_i, \qquad p \equiv -\left(\frac{\partial \epsilon}{\partial \sigma}\right)_s. \tag{1}$$

Let V_i be the velocity vector; then the *deformation* and *rotation* tensors d_{ij} and ω_{ij}, respectively, are given by the definitions

$$d_{ij} \equiv \frac{1}{2}(V_{i,j} + V_{j,i}), \qquad \omega_{ij} \equiv \frac{1}{2}(V_{i,j} - V_{j,i}). \tag{2}$$

If a secondary stress tensor W^i_j be given by the definition

$$W^i_j \equiv p\delta^i_j + T^i_j, \tag{3}$$

then the *dissipation function* Φ, given by the definition

$$\Phi = W^i_j d^j_i, \tag{4}$$